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THE GRIFFITH ENERGY PROBLEM FOR BRITTLE POLYMERS

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A number of problems concerning the Griffith criterion for brittle polymers have been considered. A general formula for the energy of deformation of a stressed sample with surface and internal cracks has been derived. This energy was calculated simultaneously for the plane and plane-deformed states of this sample. The commonly accepted view that the Griffith criterion determines a quasi-safe stress that differs markedly from the athermal-fracture criterion was verified.

In brittle materials, initial micro- and submicrocracks begin to grow to fracture cracks after these materials are subjected to the action of a load exceeding the safe load σ_0. It should be noted that microcracks differ more or less markedly by the degree of their danger, and, what is no less important, the growth of one crack leads to an unloading of the surrounding material and decelerates the growth of the neighboring cracks. Therefore, the strength and longevity of polymer materials are determined in practice by the growth of the most dangerous one or, more rarely, several micro- or submicrocracks to the critical length l_c, at which the fracture process becomes athermal (rapid) and proceeds at a critical rate v_c determined by the rate of propagation of an elastic disturbance in these materials. For organic polymers $v_c = (5–8)\cdot 10^2$ m/sec, for polymethylmethacrylate (PMMA) $v_c = (700–800)$ m/sec, and for inorganic glass $v_c = 2000$ m/sec [1]. This rate can be estimated by the Robert–Wells formula [2]

$$v_c = 0.38\sqrt{E/\rho}\,. \tag{1}$$

Investigations of submicroscopic cracks in actual polymers with the use of diffraction methods made it possible to determine their dimensions (longitudinal and transverse), form (sections in plates, disks in polymer fibers), and location (surface or internal cracks). Cracks interpreted within the framework of the mechanical models presented below had fairly small initial dimensions. The initial length l_0 of microcracks detected in samples of width (or diameter) L equal to several millimeters was 1700 Å for PMMA, 3000 Å for polyvinylbutyral, 170 Å for polyethylene, 320 Å for polypropylene, 3000 Å for polyvinylchloride, and 90 Å for caprone [2].

To this it must be added that fractography investigations of the fracture surface of polymers have shown that the critical length l_c of a crack is independent of the cross section of a sample; samples with cross sections differing by more than 100 times were used in these investigations [2]. It was established that in all cases, a fracture crack initially represents a defect of length l_0 and grows along the normal to the direction of a maximum tensile stress. Since the opening of this crack is small in comparison with its length, it can be defined as

$$\lambda << l_0 \le l(\tau) \le l_c << L\,, \quad 0 \le \tau \le \tau_f\,. \tag{2}$$

In accordance with (2), a sample shaped as a plate (or a cylindrical bead) can be interpreted as an elastic plane (x, y) with an internal crack $|x| < l_0$, $y = 0$; as an elastic semiplane $x > 0$, $|y| < \infty$ with a surface crack $0 < x < l_0$, $y = 0$; or as an elastic space (x, y, z) with an internal circular axisymmetric crack $z = 0$, $0 \le r < R_0$.

Deformation Energy of Samples with Surface and Internal Cracks. We now derive general analytical relations for calculating the deformation energy of samples representing infinite plates with surface and internal cracks.

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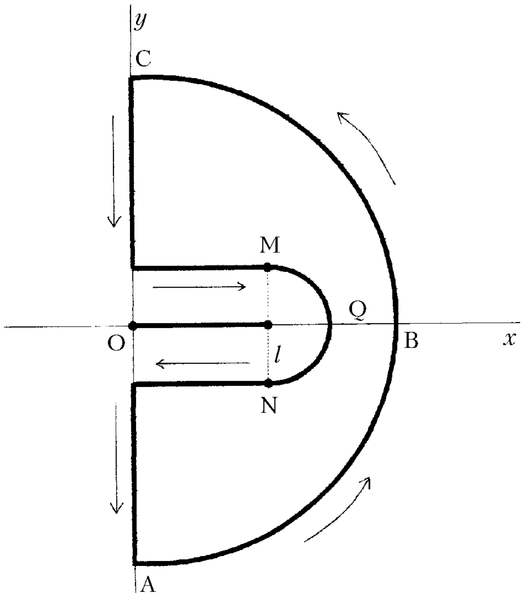

Fig. 1. On the calculation of the deformation energy of a sample with a surface crack.

In the process of loading, the forces applied to a sample do work W (equal to the deformation energy). In this case, it is assumed that the sample remains elastic and a kinetic energy is not formed. First, we will consider the case of formation of a surface crack $x < l_0$, $y = 0$ in an elastic semiplane $R^2 = (x, y)$ at $x > 0$ and $|y| < +\infty$.

Let $G^* = \{(x, y): y = 0, x < l_0\}$, $D = R^2 \setminus G^*$ be a convex region in the stressed state; $\sigma_{ij}(x, y)$, $\varepsilon_{ij}(x, y)$, and $u_i(x, y)$ be respectively components of the stress tensor, deformation tensor, and displacement vector ($i, j = x, y$). The challenge is to obtain an analytical expression for calculating the deformation energy of an elastic semiplane with a surface $x = 0$ free of stresses and a surface crack, on the edges of which an arbitrary loads act. Let us write the main statistical relations for the region D of the elastic plane (in index designations) found simultaneously in plane and plane-deformed states (these states are of independent interest):

$$\sigma_{ij,j}(x, y) = 0 , \tag{3}$$

$$\varepsilon_{ij}(x, y) = (1/2)\,[u_{i,j}(x, y) + u_{j,i}(x, y)] , \tag{4}$$

$$\sigma_{ij}(x, y) = 2\mu\varepsilon_{ij}(x, y) + \lambda^* e(x, y)\,\delta_{ij} , \tag{5}$$

where $\mu = G = E/[2(1 + \nu)]$, $\lambda^* = \nu E/(1 - \nu^2)$ for the plane state, $\lambda^* = 2G\nu/(1 - 2\nu)$ for the plane deformation, and $e(x, y) = \varepsilon_{ii}(x, y)$ ($i = x, y$). The deformation energy of the region being considered is equal to [1]

$$W = (1/2) \iint\limits_D (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{xy}\varepsilon_{xy})\, dxdy =$$

$$= (1/2) \iint\limits_D \left\{\frac{\partial}{\partial x}(\sigma_{xx}u_x + \sigma_{xy}u_y) + \frac{\partial}{\partial y}(\sigma_{yy}u_y + \sigma_{xy}u_x)\right\} dxdy , \tag{6}$$

where $\gamma_{xy} = 2\varepsilon_{xy}$. Let L_Δ be the contour formed by the lower and upper edges of the crack and C_r be the contour shown in Fig. 1 (ABC is the arc of a circle of radius r). Rewriting (6) in view of the Ostrogradskii–Gauss formula for the region restricted by the contours C_r and L_Δ, we obtain

$$W = (1/2)\int\limits_{\mathrm{ABC}}\Big[(\sigma_{xx}u_x + \sigma_{xy}u_y)\cos(n, x) + (\sigma_{yy}u_y + \sigma_{xy}u_x)\cos(n, y)\Big]\,dS +$$

$$= (1/2)\int\limits_{L_\Delta}(\sigma_{yy}u_y + \sigma_{xy}u_x)\cos(n, y)\,dS + (1/2)\int\limits_{\mathrm{MQN}}\sigma_{ij}d\varepsilon_{ij} = J_1 + J_2 + J_3\,. \tag{7}$$

Since $\cos(n, x)dS = dy = r\cos\varphi\,d\varphi$ and $\cos(n, y)\,dS = -dx = r\sin\varphi\,d\varphi$,

$$J_1 = (1/2)\int\limits_{-\pi/2}^{\pi/2}\Big[(\sigma_{xx}u_x + \sigma_{xy}u_y)\,r\cos\varphi + (\sigma_{yy}u_y + \sigma_{xy}u_x)\,r\sin\varphi\Big]\,d\varphi\,.$$

Let $|z| = |r\exp i\varphi| = r \to \infty$. Since the main vector of all the external forces acting on the region D is equal to zero, at large values of $|z|$ the stress components will be of the order of $\sigma_{ij}(x, y) = o\left(\frac{1}{|z|^2}\right)$, the quantity $u_i(x, y)$ will be limited [3], and, consequently, the condition $\lim\limits_{|z|=r\to\infty} J_1 = 0$ will be fulfilled. On MQN, $x = \varepsilon\cos\varphi$, $y = \varepsilon\sin\varphi$, and $\lim\limits_{|z|=r\to\infty} J_3 = 0$. The quantity J_2 is determined using the relation $dx = dS\cos(t', x)$ [3], where t' is the tangent to the path of integration, pointed towards the direction of displacement, as is shown in Fig. 1, and **n** is the rightward normal when looking along t'. In this case, $\cos(t', x) = 1$ and $dS = dx$ at the upper edge of the crack and $\cos(t', x) = -1$ and $dS = -dx$ at its lower edge. The above-described representation of J_2 leads to the following expression for the deformation energy of the elastic semiplane with a surface crack:

$$W = (-\,1/2)\int\limits_0^l\Big[(\overset{+}{\sigma}_{yy}\overset{+}{u}_y - \overset{-}{\sigma}_{yy}\overset{-}{u}_y) + (\overset{+}{\sigma}_{xy}\overset{+}{u}_x - \overset{-}{\sigma}_{xy}\overset{-}{u}_x)\Big]_{y=0}\,dx\,, \tag{8}$$

where the plus denots the upper edge of the crack and the minus denotes its lower edge. The deformation energy of an elastic plane (x, y) with an internal rectilinear crack $y = 0$, $|x| < l_0$ can be determined analogously:

$$W = (-\,1/2)\int\limits_{-l_0}^{l_0}\Big[(\overset{+}{\sigma}_{yy}\overset{+}{u}_y - \overset{-}{\sigma}_{yy}\overset{-}{u}_y) + (\overset{+}{\sigma}_{xy}\overset{+}{u}_x - \overset{-}{\sigma}_{xy}\overset{-}{u}_x)\Big]_{y=0}\,dx\,. \tag{9}$$

Relations (8) and (9), representing fundamental relations of brittle-fracture mechanics, can be used as general relations for the above-considered particular cases. For example, when a crack is subjected to the action of a nonuniform normal load $P(x)$, which is even with respect to x [$\sigma_{yy}^{\pm}(x, 0) = -P(x)$, $x < l_0$, $(|x| < l_0)$, $\sigma_{xy}^{\pm}(x, 0) = 0$, $x < l_0$, $(|x| < l_0)$, $u_y^{+}(x, 0) = -u_y^{-}(x, 0)$], the deformation energy of surface and internal cracks will be equal, respectively, to

$$W = \int\limits_0^{l_0} P(x)\,\overset{+}{u}_y(x, 0)\,dx\,, \quad W = 2\int\limits_0^{l_0} P(x)\,\overset{+}{u}_y(x, 0)\,dx\,. \tag{10}$$

To illustrate the second relation of (10), we will consider the problem on the extension of an elastic plane with an internal crack by a uniform stress applied at infinity, which is associated directly with the derivation of the Griffith theorem. Buckner has shown (references in [1]) that the tensile stress σ at infinity arises under the action of a load applied to the edges of a crack. Thus, the case in point is the solution of problem (3)–(5) with boundary conditions

$$\sigma_{yy}^{\pm}(x, 0) = -\sigma, \quad |x| < l_0; \tag{11}$$

$$\sigma_{xy}^{\pm}(x, 0) = 0, \quad |x| < l_0; \tag{12}$$

$$\sigma_{ij}(x, y)\Big|_{\sqrt{x^2+y^2} \to \infty} = 0, \quad i, j = x, y; \tag{13}$$

$$|u_i(x, y)| < +\infty, \quad (x, y) \cup D. \tag{14}$$

To solve this problem, we will use the Muskhelishvili–Kolosov complex potential [3]. This approach also makes it possible to investigate a more complex case — the Griffith criterion for the stress arising under the simultaneous action of mechanical and thermal loads, which is still an open question of solid-strength physics and is planned to be investigated at a later time. In [3], it was shown that the stress and displacement components in (3)–(5) can be expressed in terms of the two analytical functions $\Phi(z)$ and $\Omega(z)$ (complex potentials) of the one complex argument $z = x + iy$:

$$\sigma_{xx} + \sigma_{yy} = 4\mathrm{Re}\,[\Phi(z)], \tag{15}$$

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\,\overline{\frac{d\Phi(z)}{dz}}, \tag{16}$$

$$2G(u_x + iu_y) = \kappa\varphi_1(z) - \omega(\bar{z}) - (z - \bar{z})\,\overline{\Phi(z)}, \tag{17}$$

where

$$\varphi_1(z) = \int \Phi(z)\,dz; \quad \omega(\bar{z}) = \int \Omega(\bar{z})\,d\bar{z}; \tag{18}$$

$\kappa = (3 - \nu)/(1 + \nu)$ for the plane state and $\kappa = (3 - 4\nu)$ for the plane deformation. The indefinite integral $\int f(z)d(z)$ in (18) represents the function

$$F(z) = \int_{z_0}^{z} f(z')\,dz' + \mathrm{const}, \tag{19}$$

where the integral is taken along an arbitrary path in the holomorphy region $(x, y) \cup D$ connecting the arbitrary fixed point z_0 with the moving point z, and const is an arbitrary (complex) quantity playing no part in the further discussion. At $y = \pm 0$, from boundary conditions (11) and (12) and expression (17) we obtain the following expressions for the edges of the crack:

$$[\Phi(t) + \Omega(t)]^{+} + [\Phi(t) + \Omega(t)]^{-} = -(2\sigma) = p, \quad |t| < l_0; \tag{20}$$

$$[\Phi(t) - \Omega(t)]^{+} - [\Phi(t) - \Omega(t)]^{-} = 0 = q', \quad |t| < l_0, \tag{21}$$

which represent the boundary conditions for the linear-conjunction problems (Riemann–Hilbert problems). Their solutions have the form

$$\Phi(z) + \Omega(z) = \frac{1}{2\pi i \mathrm{X}(z)} \int_{-l_0}^{l_0} \frac{p\mathrm{X}(t)}{t - z}\, dt + \frac{2(C_0 z + C_1)}{\mathrm{X}(z)}\,, \tag{22}$$

$$\Phi(z) - \Omega(z) = \frac{1}{2\pi i} \int_{-l_0}^{l_0} \frac{q' dt}{t - z} + [\Phi(\infty) - \Omega(\infty)]\,, \tag{23}$$

where C_0 and C_1 are indeterminate constants that should be determined;

$$\mathrm{X}(z) = \sqrt{z^2 - l_0^2}\,; \tag{24}$$

$X(t) = X^+(t) = X(z)\big|_{y=+0} = \sqrt{t^2 - l_0^2}$ is the value of this function at the upper edge of the crack and $X^-(t) = X(z)\big|_{y=-0} = \sqrt{t^2 - l_0^2}$ is the value of the function at the lower edge of the crack. In this case,

$$\sqrt{z^2 - l_0^2}\,\Big|_{y=\pm 0} = \pm\, i \sqrt{l_0^2 - t^2}\,, \quad |t| < l_0\,. \tag{25}$$

The asymptotic behavior of the complex potentials at infinity leads to the following relations:

$$\Phi(\infty) - \Omega(\infty) = -\overline{\Gamma}', \quad \Omega(\infty) = \overline{\Gamma} + \overline{\Gamma}', \quad \Gamma' = (1/2)\left(\sigma_{yy}^{(\infty)} - \sigma_{xx}^{(\infty)}\right) + i\sigma_{xy}^{(\infty)}\,,$$

$$\Gamma' = (1/2)\left(\sigma_{yy}^{(\infty)} - \sigma_{xx}^{(\infty)}\right) - i\sigma_{xy}^{(\infty)}\,, \quad \Gamma = (1/2)\left(\sigma_{yy}^{(\infty)} + \sigma_{xx}^{(\infty)}\right) + iC'\,, \quad C' = 2G\varepsilon_\infty/(1 + \kappa)\,,$$

where ε_∞ is a constant determining the rotation of the infinitely distant part of the plane and having no influence on the stress. The quantity C' can be assumed to be equal to zero, and it follows from conditions (13) that $\Gamma' = \Gamma' = 0$ and $\Gamma = \Gamma = 0$. Thus, $\Omega(\infty) = 0$, $\Phi(\infty) = 0$, and, according to (22) and (23),

$$\Phi(z) = \Omega(z) = (-\sigma/2)\left(1 - \frac{z}{\sqrt{z^2 - l_0^2}}\right) + \frac{C_0 z + C_1}{\sqrt{z^2 - l_0^2}}\,. \tag{26}$$

Since $\Phi(\infty) = \lim\limits_{|z| \to \infty} \Phi(z) = 0$, it follows from (26) that $C_0 = \Phi(\infty) = 0$. The coefficient C_1 is determined on condition that the displacement be univalent. According to (17), this condition is fulfilled if the expression $\kappa\varphi_1(z) - \omega(\overline{z})$ takes its initial value when the point z traces out the closed contour Λ around the crack. Contracting the contour Λ to the segment L_Δ formed by the upper and lower edges of the crack, we obtain the condition for the constant C_1:

$$\kappa \int_{-l_0}^{l_0} \left[\Phi^+(t) - \Phi^-(t)\right] dt + \int_{-l_0}^{l_0} \left[\Omega^+(t) - \Omega^-(t)\right] dt = 0\,,$$

from which and from expressions (25) and (26) it follows that $C_1 = 0$. Thus,

$$\Phi(z) = \Omega(z) = (-\sigma/2)\left(1 - \frac{z}{\sqrt{z^2 - l_0^2}}\right). \tag{27}$$

For the case described by (11)–(14), the deformation energy is equal, according to (10), to

$$W = 2\sigma \int_0^{l_0} u_y^+(x, 0)\, dx\,. \tag{28}$$

The displacement component in (28) is determined from (17):

$$u_y(x, y)\big|_{y=+0} = \frac{1}{2G}\left\{\mathrm{Im}\,[\kappa\varphi_1(z) - \omega(\bar{z})]\right\}_{y=+0} = \frac{\sigma(\kappa+1)}{4G}\sqrt{l_0^2 - x^2}\,,$$

whence, we obtain, using (28), the desired quantity

$$W = \frac{\pi\sigma^2 l_0^2(\kappa+1)}{8G}\,. \tag{29}$$

Expression (29) forms the basis for the Griffith theory [4], in which the development of a crack in a body is explained by the accumulation and release of a deformation energy in this body. As a model, Griffith used an elastic body containing a crack (displacement-discontinuity surface) of area S'. When the displacement-discontinuity surface increases by a virtual value $\delta S'$, external forces applied to the body do a work δA_S equal to the energy δW by which the elastic energy of the body decreases as a result of the action of these forces. On the other hand, an increase in the surface of the crack leads to an increase in its surface energy by $\delta\Pi$. According to Griffith, the energy criterion of the equilibrium state of a crack is $\delta(W - \Pi) = 0$. Thus, a crack is in the stable state at $\delta\Pi > \delta W$, in the nonstable state at $\delta\Pi < \delta W$, and in the neutral (equilibrium) state at $\delta\Pi = \delta W$. In the case described by (11)–(14), the parameters of the critical equilibrium state are determined from the expression $\delta(W - \Pi)/dl_0 = 0$, where $\Pi = 4\alpha_{\mathrm{sur}} l_0$ (α_{sur} is the free surface energy of a material in vacuum). Hence, the Griffith criterion has the form

$$\sigma_{\mathrm{Gr}} = 4\sqrt{\frac{\alpha_{\mathrm{sur}} G}{\pi(\kappa+1)\,l_0}} \tag{30}$$

for the plane state and the plane deformation. At $\sigma > \sigma_{\mathrm{Gr}}$, a crack increases at a rate determined by (1), and at $\sigma < \sigma_{\mathrm{Gr}}$ the crack does not increase. Thus, the threshold stress σ_{Gr} is, according to Griffith, the fracture criterion. Up till now, Griffith's standpoint has been adopted by many researchers considering this energy approach as an important tool for the derivation of theoretical relations for different conditions of fracture of brittle solid bodies [5].

The first doubts of the reliability of the physical meaning of the Griffith criterion were cast when relation (30) (obtained by Griffith too) was used as a basis of the fractometry method of determining an important strength characteristic of a material in the plane — the free energy of its surface:

$$\sigma_{\mathrm{Gr}} = \sqrt{\frac{2E\alpha_{\mathrm{sur}}}{\pi l_0}}\,. \tag{31}$$

The quantity α_{sur} is calculated for a microcrack of length l_0 arising on the surface of a sample under the action of the critical stress σ_{c} breaking the sample; in this case, it is assumed that $\sigma_{\mathrm{c}} = \sigma_{\mathrm{Gr}}$. However, the values of α_{sur} calculated by (31) were very overstated. For example, the calculations of α_{sur} for inorganic glass in vacuum, performed by Berdennikov (references in [2]), gave $\alpha_{\mathrm{sur}} = 1.2$ J/m^2, while the experimental value of this quantity, obtained by Tsisman [1], is equal to $\alpha_{\mathrm{sur}} = 0.5$ J/m^2. This circumstance can be explained by two reasons. In order that (31) could be used, it was assumed that the length of an edge crack is two times smaller than the length of an internal crack. However, this assumption is incorrect since, in the case where σ_{Gr} for an edge microcrack is strictly calculated from the relation $\delta(W - \Pi)/dl_0 = 0$, the deformation energy W should be determined with account for the influence of the free boundary of the body. This problem was considered in [6], where it was shown that the deformation energy of a sample with a surface crack $y = 0$, $0 < x < l_0$ formed as a result of the extension of the sample by a constant stress σ is equal to

$$W = \frac{\pi\sigma^2 l_0^2}{2E}\,, \tag{32}$$

and is two times smaller than the deformation energy necessary for the formation of an internal crack $y = 0$, $|x| < l_0$. If the surface energy of an edge crack $\Pi = 2\alpha_{\mathrm{sur}} l_0$, the Griffith energy criterion leads to the following expression for the threshold stress of this crack:

$$\sigma_{\mathrm{Gr}} = \sqrt{\frac{2E\alpha_{\mathrm{sur}}}{\pi l_0}} \qquad (33)$$

This stress is equal to the stress determined by (31) for an internal crack. Result (33) is of fundamental importance for solid-strength physics; it was predicted earlier in [2] in the process of investigating the thermokinetics of the brittle fracture of polymers in mechanical fields.

The second reason for the above-indicated difference is that, in actuality, the fracture threshold σ_{Gr} differs from the critical stress σ_{c}: the experimentally determined stress σ_{c} breaking a sample differs from the theoretical fracture threshold σ_{Gr} obtained in calculating α_{sur} in (31). According to the notions of the theory of thermofluctuation strength [2, 7], the critical stress σ_{c} responsible for the athermal stage of the fracture process proceeding at a rate determined by (1) is equal to

$$\sigma_{\mathrm{c}} = \frac{(U_0 - qT)\sqrt{\lambda}}{\chi V_a} l_0^{-1/2}, \qquad (34)$$

where U_0 is the energy of activation of an elementary act of fracture in the absence of a stress (this energy is close in value to the chemical-bond energy of polymers) and $\chi = 0.79$ and 0.71 for the surface and internal cracks respectively. The first considerations concerning the physical meaning of criterion (31) were made in [7] on the basis of the thermodynamic approach to the fracture process; it was suggested that σ_{Gr} is not a fracture criterion in the commonly accepted sense and is equal to the safe stress. These considerations will be supported below by strict quantitative calculations performed on the basis of an analytical formula for the rate of growth of a crack. Thermofluctuational fracture processes can be described at the atomic-molecular level with the use of the model of weakly coupled harmonic oscillators. In this model, an elementary act of breaking (or restoration) of the chemical bonds at the top of a crack is interpreted as the classical climb over the potential barrier [1]. This made it possible to obtain, using the molecular model of a crack, an analytical expression for the frequency of breaking and restoration of the chemical bonds at the top of the crack [1]. Then, using the thermodynamic potentials for describing the brittle fracture of polymers on the basis of physical kinetics accounting for the statistics of breaking and recombinations of the bonds at the top of a crack, occurring with a definite frequency, we obtain the following generalized expression for the rate of growth of the crack with account for the deformation energy of the sample:

$$v\,(l, \sigma, T) = 2\lambda \nu_0 \exp\left(-\frac{U_0 - qT - \alpha_{\mathrm{sur}}\lambda\lambda_\pi}{kT}\right)\sinh\left(\frac{V_a\sigma^* - \alpha_{\mathrm{sur}}\lambda\lambda_\pi}{kT} + \frac{\pi\sigma^2 l_0\lambda\lambda_\pi}{2EkT}\right). \qquad (35)$$

Here, $\sigma^* = \sigma\beta(l_0)\sqrt{l/l_0}$ (in the volume V_a) and $\beta(l_0) = \chi\sqrt{l_0/\lambda}$ is the coefficient of stress concentration at the top of the crack, corresponding to its initial length; the other parameters will be described below. At any external stress $\sigma = \sigma_{\mathrm{Gr}}^{(0)}$, between the processes of breaking and restoration of chemical bonds, there arises a dynamic-equilibrium state at the crack top. In this case, the rate of growth of the crack is equal to $v(l_0, \sigma_{\mathrm{Gr}}^{(0)}, T) = 0$. The stress $\sigma_{\mathrm{Gr}}^{(0)}$ corresponding to the equilibrium state will be called the quasi-safe stress [8]. The value of $\sigma_{\mathrm{Gr}}^{(0)}$ will be determined from (35) at $v(l_0, \sigma_{\mathrm{Gr}}^{(0)}, T) = 0$ with allowance for the fact that, in this case, the local stress $\sigma = \sigma_0^* = \beta\sigma_{\mathrm{Gr}}^{(0)}$:

$$\sigma_{\mathrm{Gr}}^{(0)} = (\beta\lambda_m E/\pi l_0)\left[\sqrt{1 + 2\pi l_0\alpha_{\mathrm{sur}}/(\beta^2\lambda_m E)} - 1\right]. \qquad (36)$$

To simplify relation (36), we will estimate the term $W = 2\pi l_0\alpha_{\mathrm{sur}}/(\beta^2\lambda_m E)$ for a number of polymers (organic and inorganic). For example, for PMMA [2] $l_0 = 0.23$ μm, $\alpha_{\mathrm{sur}} = 39\cdot10^{-3}$ J/m^2, $\beta = 9$, $\lambda_m = 1.5\cdot10^{-4}$ μm, and $E = 3.93\cdot10^9$ N/m^2, and for inorganic (silicate) glass [2] $l_0 = 4$ μm, $\alpha_{\mathrm{sur}} = 0.5$ J/m^2, $\beta = 60$, $\lambda_m = 1.64\cdot10^{-4}$ μm, and $E = 5.89\cdot10^{10}$ N/m^2. In both cases, $W >> 1$; therefore, simplifying (36), we find that

$$\sigma_{\mathrm{Gr}}^{(0)} = \sqrt{\frac{2E\alpha_{\mathrm{sur}}}{\pi l_0}}, \qquad (37)$$

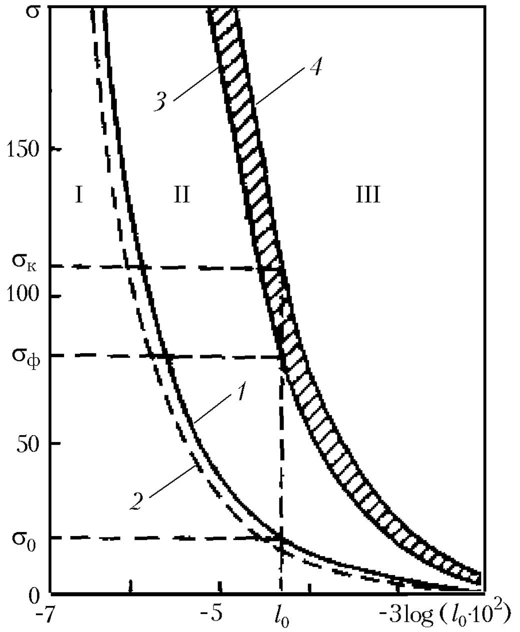

Fig. 2. Diagram of the strength states of the PMMA with a surface crack at 20°C: 1) σ_0 (log l_0); 2) σ_{Gr} (log l_0); 3) boundary of the transient region; 4) σ_c (log l_0). σ, MPa; l_0, m.

This quantity is equal to the Griffith fracture threshold. Consequently, the Griffith criterion $\sigma_{Gr}^{(0)}$ determines the stress beginning with which a crack grows with an increasing rate in accordance with (35). A stress lower than $\sigma_{Gr}^{(0)}$ can also lead to a fracture: according to [8], a crack formed due to thermal fluctuations can attain, in a random way, a state in which the rate of its growth becomes positive. In the experiments on the long-term strength of the polymers being considered, at stresses smaller than $\sigma_{Gr}^{(0)}$, the longevity isotherm grew to the region of infinitely large values: $\log \tau(\sigma < \sigma_{Gr}^{(0)}, T) \to \infty$.

The problem on the physical meaning of the Griffith criterion can be considered in a different context — on the basis of the so-called diagram of strength states (σ, l_0) of polymer materials. In the thermodynamic fluctuation theory of strength [7], another critical state is introduced — the safe stress σ_0, characterizing the asymptotical approach of the longevity isotherm to larger times: $\log \tau(\sigma, T) \to \infty$ at $\sigma \to \sigma_0$. According to [2], the safe stress is equal to

$$\sigma_0 = \frac{\alpha_{\text{sur}}}{\beta\lambda_m} = \frac{\alpha_{\text{sur}}\sqrt{\lambda}}{\chi\lambda_m} l_0^{-1/2} . \tag{38}$$

The value of σ_0 is also calculated from relation (35) but without account for the influence of the deformation energy of a stressed sample on the growth of a crack; in this case, the energy balance between the elastic energy of an individual chemical bond (or a group of bonds) at the top of the crack and the energy of the fracture surface formed is considered. Figure 2 shows the diagram of strength states, important in practice, of PMMA samples shaped as strips with edge microcracks of different length: curve 1 corresponds to the safe stress σ_0 determined by (38), curve 2 corresponds to the quasi-safe stress $\sigma_{Gr}^{(0)}$ determined by (37), curve 4 corresponds to the critical stress determined by (34), and curve 3 corresponds to the boundary of the transient region $\sigma_f < \sigma < \sigma_c$, where the athermal mechanism begins to manifest itself and the contribution of this mechanism to the total fracture mechanism becomes comparable to the contribution of the thermofluctuation mechanism.

The value of σ_f is calculated from the relation $\tau_f(\sigma_f, T) = \tau_c$, where $\tau_f(\sigma_f, T)$ is the longevity of a sample at the thermofluctuation-fracture stage [2] and τ_c is the time of the athermal stage of the fracture process proceeding at a rate determined from (1): $\tau_c \sim (L - l_c)/v_c$. It follows from the indicated diagram that the Griffith threshold stress $\sigma_{Gr}^{(0)}$ is practically equal to the safe stress σ_0 and the quantity $\sigma_{Gr}^{(0)}$ is not a criterion of the athermal fracture defined by σ_c, because σ_c is an order of magnitude larger than this quantity. Moreover, the change to the athermal-fracture mechanism happens earlier than σ_c is attained. From the physical standpoint, it is more correct to consider the quantity σ_f, which is somewhat smaller than σ_c, as the critical stress. When the temperature decreases, σ_f tends to σ_c and, at absolute zero, these quantities become equal. The diagram presented also makes it possible to determine the degree of danger of surface (or internal) defects at a definite external load and to classify the regions of initial defects with respect to the stresses under which they are formed: I) region of submicronic cracks (quasi-safe stresses $\sigma_{\text{Gr}}^{(0)}$, at which the longevity of a sample can be infinitely large); II) region of microscopic cracks (an athermofluctuation fracture occurring at a rate determined by (35); III) region of macrocracks (an athermal fracture occurring at a rate determined by (1)). The mechanism of fracture of a sample depends on the region in which a point of an initial crack having a definite length is found. In other words, in polymers there are a number of sub- and microcracks distributed depending on the degree of their danger, which was supported experimentally in [2] in the process of investigating the statical strength and longevity properties of polymer films and fibers on the basis of the corresponding levels of their strength characteristics. In [2], the discrete character of the defect structure of polymers was determined in the process of investigating the function of distribution of the strength and longevity of their stressed samples. Our consideration will be incomplete if we do not explain the Griffith experiment in which the condition $\sigma_{\text{Gr}}\sqrt{l_0}$ = const was verified. Griffith, in his experiments, made cracks of different length at the edge of a glass plate and measured, under the conditions of one-axis extension, the load at which the glass was broken down. He found that $\sigma_{\text{Gr}}\sqrt{l_0} = 25.5\cdot10^4$ N/m$^{3/2}$. Relation (35) makes it possible to explain the Griffith experiment. In the case where the stresses in (35) are much larger than the quasi-safe stress, the recombination of bonds at the top of a crack can be disregarded and

$$v\,(l, \sigma, T) = \lambda v_0 \exp\left(-\frac{U_0 - qT - V_a\sigma^* - \pi\sigma^2 l_0\lambda\lambda_\pi/2E}{kT}\right). \tag{39}$$

From this relation we obtain the following criterion for the change to the athermal stage of the fracture: when a crack attains the critical length $l = l_c$, the local stress at its top reaches the critical value $\sigma_c^* = \beta\sigma\sqrt{l_c/l_0}$ and the rate of growth of the crack becomes constant independently of the temperature. The latter means that $U_0 - qT - V_a\sigma_c^* - \pi^2 l_c\lambda\lambda_\pi/(2E) = 0$. We will solve this equation with respect to the quantity $\sqrt{l_c/l_0}$ and estimate the relative critical length of the crack (as in (36)) $\sqrt{l_c/l_0}$. For PMMA [2]: $\lambda = 12\cdot10^{-4}$ μm, $\lambda_\pi = 8\cdot10^{-4}$ μm, $U_0 = 1.33\cdot10^5$ J/mole, $q = 16.76\cdot10^5$ J/(mole·K), $T = 293$ K, and $V_a = 1.4\cdot10^{-28}$ m^3; for inorganic (silicate) glass [2]: $\lambda = 5.4\cdot10^{-4}$ μm, $\lambda_\pi = 10.8\cdot10^{-4}$ μm, $U_0 = 5.66\cdot10^5$ J/mole, $q = 41.9\cdot10^5$ J/(mole·K), $T = 293$ K, and $V_a = 9.6\cdot10^{-29}$ m^3 (the other constants are presented above). Both variants give

$$\sqrt{l_c/l_0} = \sqrt{\frac{2E\,(U_0 - qT)}{\pi\sigma^2 l_0\lambda\lambda_\pi}}\ . \tag{40}$$

In the case where a critical (breaking) stress $\sigma = \sigma_c$ is applied to a sample, $l_0 = l_c$ and expression (40) leads to the relation

$$\sigma_c = \sqrt{\frac{2E\,(U_0 - qT)}{\pi l_0\lambda\lambda_\pi}}\ , \tag{41}$$

explaining the sense of the Griffith experiment. Actually, it follows from (41) that $\sigma_c\sqrt{l_0}$ = const and, for the inorganic glass with the above-indicated constants we obtain from (41) that $\sigma_c\sqrt{l_0} = 24.5\cdot10^4$ N/m$^{3/2}$, which is practically equal to the value obtained in the Griffith experiments.

Thus, the condition $\sigma_c\sqrt{l_0}$ = const verified experimentally by Griffith corresponds to the result obtained using (41). However, Griffith did not obtain this relation because he used the mechanical approach. Relation (41) also leads to another interesting result. According to [1, 2], the fluctuation volume $V_a = \lambda\lambda_\pi\lambda_m$, $\sigma_0 = \alpha_{\text{sur}}/(\beta\lambda_m)$, the structure-sensitive coefficient in the Zhurkov longevity formula $\gamma = V_a\beta$, and expression (41) takes the form

$$\sigma_c = \sigma_{\text{Gr}}\sqrt{(U_0 - qT)/(\gamma\sigma_0)}\ , \tag{42}$$

from which the difference between the Griffith damage threshold (in the generally accepted terminology) and the critical (breaking) stress is evident. Up to this point, in the majority of scientific publications on the strength and fracture of polymer materials, the quantity σ_{Gr} has been considered as the criterion of athermal fracture. The relations presented above show that this standpoint should be revised because the Griffith criterion is not a fracture criterion in the generally accepted sense.

NOTATION

A_S, work of the external forces; C_r, contour (Fig. 1); C_0, C_1, C', constants; D, convex region outside the crack; E, Young modulus; e, sum of deformations along the axis on a plane; f and F, functions; G, shear modulus; G^*, region consisting of points of a crack; J_1, J_2, J_3, integrals; Im, imaginary part; k, Boltzmann constant; l, l_0, and l_c, current, initial, and critical length of the crack; L, length of a sample shaped as a strip; L_Δ, contour formed by the lower and upper edges of the crack; $\mathbf{n}$, normal to the contour L_Δ; $P(x)$, normal load on the crack; p, sum of functions; Π, surface energy of the crack; q', algebraic sum of functions; q, coefficient of temperature dependence of the activation energy; r, radius of a circle; R_0, initial radius of a circular crack; R^2, points of the plane; Re, real part; dS, differential of the arc S; S', area of the crack; t', tangent to the contour of the crack; T, temperature; t, variable of integration along the length of the crack; U_0, activation energy in the absence of a stress; u_i, components of the displacement vector; $u_{i,j}$, derivative of the displacement component; $u_y^{\pm}$, displacement component at the upper and lower edges of the crack, respectively; V_a, fluctuation volume at the top of the crack; v, rate of growth of the crack; v_c, critical rate; W, deformation energy (elastic energy of a body); x, y, Cartesian coordinates; $z = x + iy$, $\overline{z} = x - iy$, complex numbers; z', integration variable; α_{sur}, free surface energy; β, coefficient of stress concentration; γ, structure-sensitive coefficient; γ_{xy}, deformation; Γ, Γ', $\overline{\Gamma}$, $\overline{\Gamma}'$, constants; δ_{ij}, Kronecker symbol; ε_{ij}, deformation; ε_∞, constant; ε, radius of semicircle MQN (Fig. 1); κ, constant; λ^*, Lamé coefficient; λ, fluctuation extension of the crack; Λ, closed (virtual) contour enclosing the crack; λ_i, λ_m, components of the fluctuation volume; μ, isothermal Lamé coefficient (equal to the shear modulus); ν, Poisson coefficient; ν_0, frequency of thermal fluctuations of kinetic units; ρ, density; σ_{ij}, stresses; σ, constant tensile stress; σ_c, critical (external) stress; σ_f, the smallest stress of the transient region; σ^*, local stress at the top of the crack; σ_{Gr}, Griffith stress; $\sigma_{\text{Gr}}^{(0)}$, quasi-safe Griffith stress; $\sigma_{yy}^{(\infty)}$, $\sigma_{xx}^{(\infty)}$, $\sigma_{xy}^{(\infty)}$, components of the stress tensor at infinitely distant points; τ, time; τ_f, longevity at the fluctuation stage of fracture; τ_c, longevity at the athermal stage of fracture; $\Phi(z)$, $\varphi_1(z)$, functions; φ, polar angle; χ, constant; $X(z)$, Plemel function; $\Omega(z)$, $\omega(z)$, functions. Subscripts: c, critical; sur, surface; f, fluctuation; $'$, variable.

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